Straight Lines and Gradients

At GCSE you met the equation of a straight line

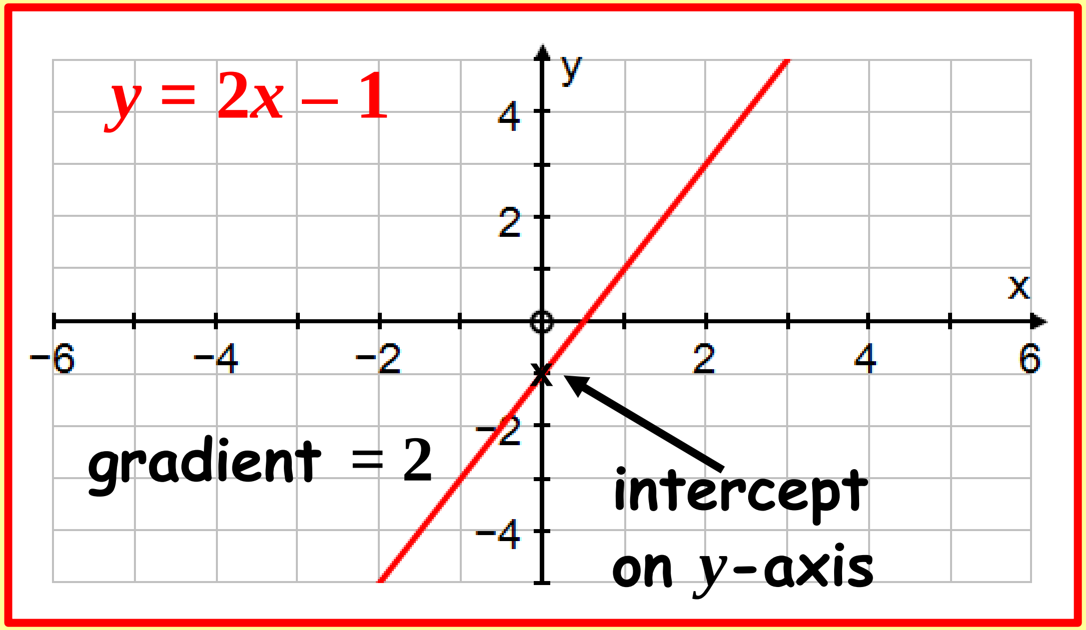
$$y = mx + c$$

m is the gradient of the line

c is the y-coordinate of the point where the line meets the y-axis, the y-intercept.

e.g. $y = 2x - 1$ has gradient $m = 2$

and y-intercept $c = -1$



Straight Lines and Gradients

Just before we start . . . What is wrong with this?

I want to simplify $(x-3)^2+2$ and I write . . .

$$(x-3)^2 = x^2-6x+9+2 = x^2-6x+11$$

The answer is correct but equals here is not right!

To avoid this sort of error I keep to equations having only two sides and instead, if I need two stages for a calculation, I would write

$$(x-3)^2+2 = x^2-6x+9+2$$

$$\Rightarrow \quad (x-3)^2+2 = x^2-6x+11$$

I use this symbol a great deal. It means if the statement before it is true, the next one is also true. We read it as implies.

Straight Lines and Gradients

If we want to find the equation of a line we need its gradient.

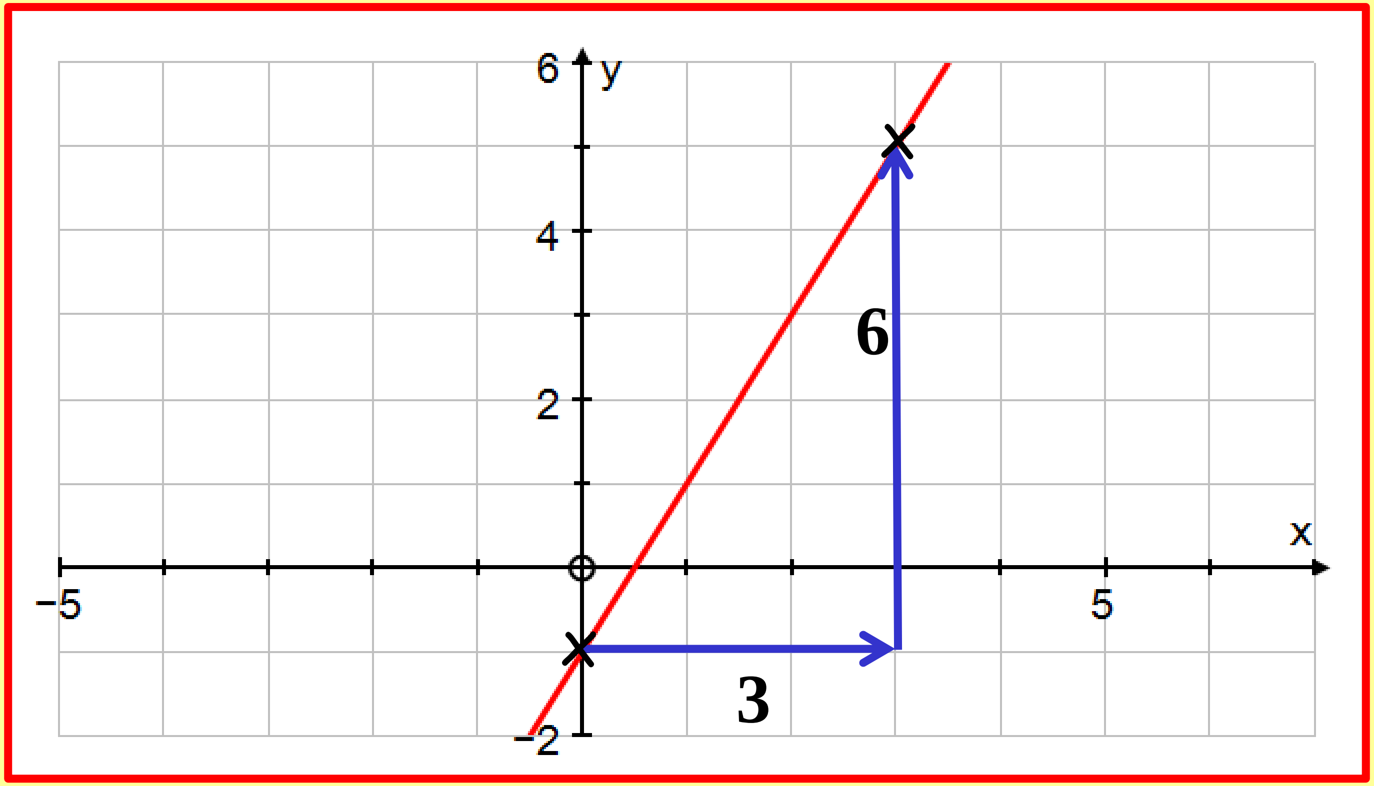
If we don't know the gradient, we have to find it using two points on the line.

To do this, we can use a formula.

We develop the formula by reminding ourselves about the meaning of a gradient.

Straight Lines and Gradients

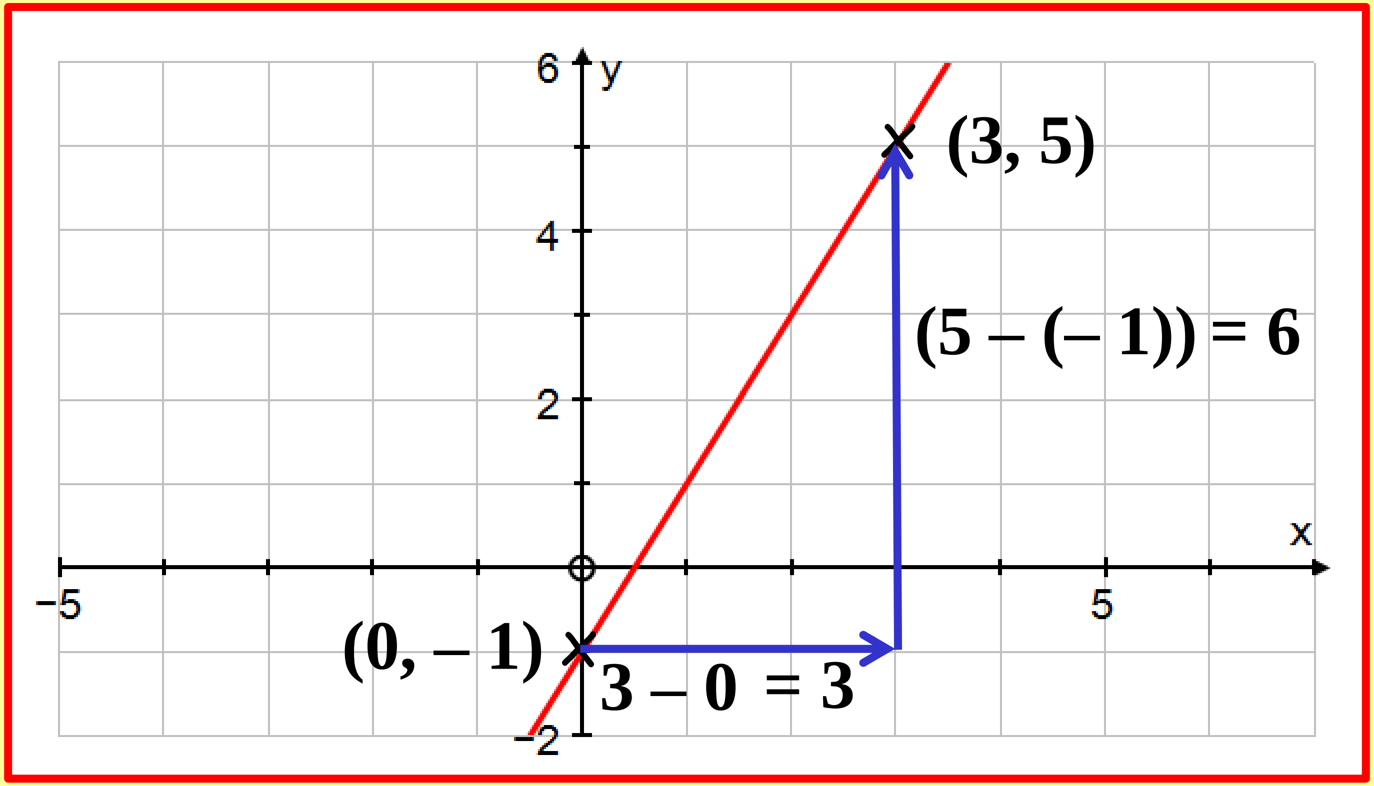
e.g.



$$m = \frac{6}{3} \Rightarrow m = 2$$

Straight Lines and Gradients

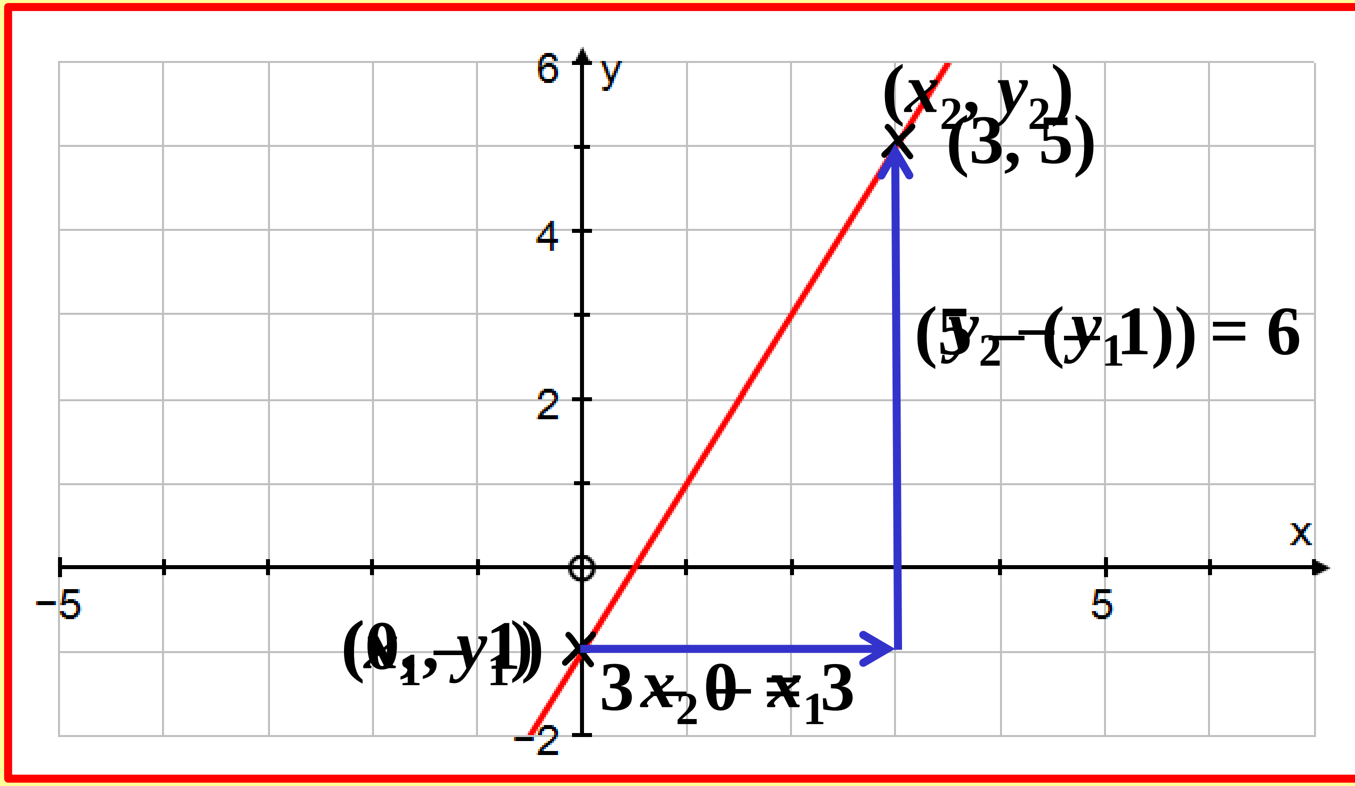
e.g.



$$m = \frac{5-(-1)}{3-0} \Rightarrow m = \frac{6}{3} \Rightarrow m = 2$$

Straight Lines and Gradients

e.g.



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Straight Lines and Gradients

The gradient of the straight line joining the points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

To use this formula we don't need a diagram.

e.g. Find the gradient of the straight line joining the points $(0, -1)$ and $(2, 3)$.

(x_1, y_1) (x_2, y_2)

Solution:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow \quad m = \frac{3 - (-1)}{2 - 0}$$

$$\Rightarrow \quad m = \frac{4}{2} \quad \Rightarrow \quad m = 2$$

A Second Formula for a Straight Line

Suppose now that (x_1, y_1) is a fixed point on the line but (x, y) is any point on the line.

We know that the gradient of the line is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Replacing (x_2, y_2) by (x, y) $\Rightarrow$ $m = \dfrac{y - y_1}{x - x_1}$

Multiplying by $(x - x_1)$: $m(x - x_1) = y - y_1$

So,

$$y - y_1 = m(x - x_1)$$

Straight Lines and Gradients

$$y - y_1 = m(x - x_1)$$

We don't <u>have</u> to use this second form of the equation of a straight line.

However, exam papers show that students make fewer mistakes in writing down the equation so gain marks (even if they make a mistake simplifying it).

From now on I will mostly use this form although the final answers will be without the brackets.

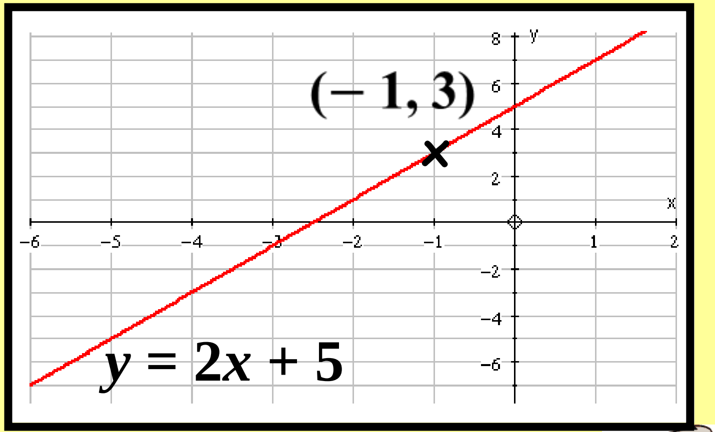
Straight Lines and Gradients

To find the equation of a straight line we need
either: one point on the line and the gradient
or: two points on the line

e.g. Find the equation of the line with gradient 2 passing through the point $(-1, 3)$.

Solution: $y - y_1 = m(x - x_1)$

$y_1 = 3$, $m = 2$ and $x_1 = -1$



Substituting in the formula:

$$\Rightarrow \quad y - 3 = 2(x - (-1))$$

$$\Rightarrow \quad y - 3 = 2x + 2$$

$$\Rightarrow \quad y = 2x + 5$$

Straight Lines and Gradients

Find the equation of the line through the points $(2, -3)$ and $(-1, 3)$.

Solution: $y - y_1 = m(x - x_1)$ We need m.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where $x_1 = 2$ and $y_1 = -3$.

$x_2 = -1$ and $y_2 = 3$.

$$m = \frac{3 - (-3)}{-1 - 2} \Rightarrow m = \frac{6}{-3} \Rightarrow m = -2$$

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - (-3) = -2(x - 2)$$

$$\Rightarrow y + 3 = -2x + 4 \Rightarrow y = -2x + 1$$

We could use the 2nd point (-1, 3) instead of (2, -3).

Straight Lines and Gradients

SUMMARY

- Gradient, m, of a straight line is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where (x_1, y_1) and (x_2, y_2) are points on the line.

- Equation of a straight line is

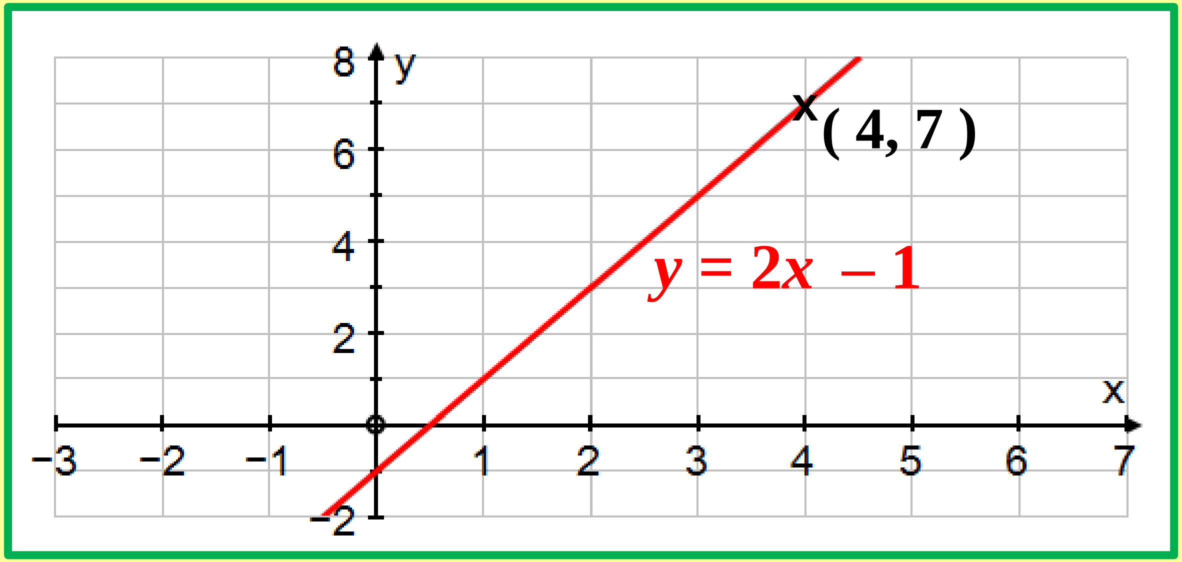
$$y - y_1 = m(x - x_1)$$

Where m is the gradient and (x_1, y_1) is a point on the line.

Straight Lines and Gradients

- The coordinates of any point lying on a line **satisfy** the equation of the line

e.g.



Substituting $x = 4$ in $y = 2x - 1$ gives

$$y = 2(4) - 1 \quad \Rightarrow \quad y = 7$$

$y = 7$ is the y-coordinate of the point (4,7) showing that the point lies on the line.

Exercise

1. Find the equation of the line with gradient 2 which passes through the point $(4, -1)$.

Solution: $y - y_1 = m(x - x_1) \Rightarrow y + 1 = 2(x - 4)$

$\Rightarrow y + 1 = 2x - 8$

$\Rightarrow y = 2x - 9$

2. Find the equation of the line through the points $(-1, 2)$ and $(1, -4)$

Solution: $m = \dfrac{y_2 - y_1}{x_2 - x_1} \Rightarrow m = \dfrac{-6}{2} \Rightarrow m = -3$

$y - y_1 = m(x - x_1) \Rightarrow y - 2 = -3(x - (-1))$

$\Rightarrow y - 2 = -3x - 3 \Rightarrow y = -3x - 1$

Straight Lines and Gradients

We sometimes rearrange the equation of a straight line so that zero is on the right-hand side (r.h.s.)

e.g. $y = -2x + 1$ can be written as $2x + y - 1 = 0$

We must take care with the equation in this form.

e.g. Find the gradient of the line with equation

$$4x - 3y + 7 = 0$$

Solution: Rearranging to the form $y = mx + c$

e.g. $4x - 3y + 7 = 0 \quad \Rightarrow \quad -3y = -4x - 7$

Dividing by -3: $\quad y = \dfrac{-4x}{-3} - \dfrac{7}{-3} \quad \Rightarrow \quad y = \dfrac{4}{3}x + \dfrac{7}{3}$

So, the gradient is given by $m = \dfrac{4}{3}$

Straight Lines and Gradients

➢ Parallel and Perpendicular Lines

If 2 lines have gradients m_1 and m_2, then:

- They are parallel if $m_2 = m_1$
- They are perpendicular if $m_2 = -\frac{1}{m_1}$

The perpendicular formula can be written as

$$m_1 \times m_2 = -1$$

(A proof of the formula for the gradients of perpendicular lines is at the end of this presentation)

Straight Lines and Gradients

Find the equation of the line parallel to $y = 2x + 1$
which passes through the point $(-1, -3)$

Solution: The given line has gradient 2. Let $m_1 = 2$

For parallel lines, $m_2 = m_1$ $\Rightarrow$ $m_2 = 2$

$y - y_1 = m(x - x_1)$ $\Rightarrow$ $y - y_1 = 2(x - x_1)$

$(-1, -3)$ on the line $\Rightarrow$ $y - (-3) = 2(x - (-1))$

$\Rightarrow$ $y + 3 = 2(x + 1)$

$\Rightarrow$ $y = 2x - 1$

Straight Lines and Gradients

e.g. Find the equation of the line perpendicular to $y = 2x + 1$ passing through the point $(1, 4)$

Solution: The given line has gradient 2. Let $m_1 = 2$

Perpendicular lines: $m_2 = -\frac{1}{m_1}$ $\Rightarrow$ $m_2 = -\frac{1}{2}$

$y - y_1 = m(x - x_1)$ $\Rightarrow$ $y - y_1 = -\frac{1}{2}(x - x_1)$

$(1, 4)$ on the line $\Rightarrow$ $y - 4 = -\frac{1}{2}(x - 1)$

Multiply by 2: $\Rightarrow$ $2y - 8 = -(x - 1)$

$\Rightarrow$ $2y - 8 = -x + 1$

$\Rightarrow$ $2y = -x + 9$

Straight Lines and Gradients

SUMMARY

➢ **Method of finding the equation of a straight line:**

- If the gradient isn't given, find the gradient using

  either parallel lines: $m_2 = m_1$

  or perpendicular lines: $m_2 = -\frac{1}{m_1}$

  or 2 points on the line: $m = \frac{y_2 - y_1}{x_2 - x_1}$

- Substitute into $y - y_1 = m(x - x_1)$

Exercise

1. Find the equation of the line parallel to the line $y + 2x - 1 = 0$ which passes through the point $(1, -3)$

2. Find the equation of the line through the point $(1, 2)$, perpendicular to the line $2y + x + 3 = 0$

Straight Lines and Gradients

Solution:

1. $y + 2x - 1 = 0 \quad \Rightarrow \quad y = -2x + 1 \quad \Rightarrow \quad m = -2$

$y - y_1 = m(x - x_1) \quad \Rightarrow \quad y - y_1 = -2(x - x_1)$

$(1, -3)$ on the line $\quad \Rightarrow \quad y - (-3) = -2(x - 1)$

$\Rightarrow \quad y + 3 = -2x + 2$

$\Rightarrow \quad y = -2x - 1$

2. $2y + x + 3 = 0 \quad \Rightarrow \quad y = \frac{1}{2}(-x - 3) \quad \Rightarrow \quad m_1 = -\frac{1}{2}$

$m_2 = -\frac{1}{m_1} \quad \Rightarrow \quad m_2 = 2.$ Also, $(1, 2)$ is on the line

$y - y_1 = m(x - x_1) \quad \Rightarrow \quad y - 2 = 2(x - 1)$

$\Rightarrow \quad y = 2x$

Straight Lines and Gradients

A proof follows for the result that if 2 lines with gradients m_1 and m_2 are perpendicular, then

$$m_2 = -\frac{1}{m_1}$$

Straight Lines and Gradients

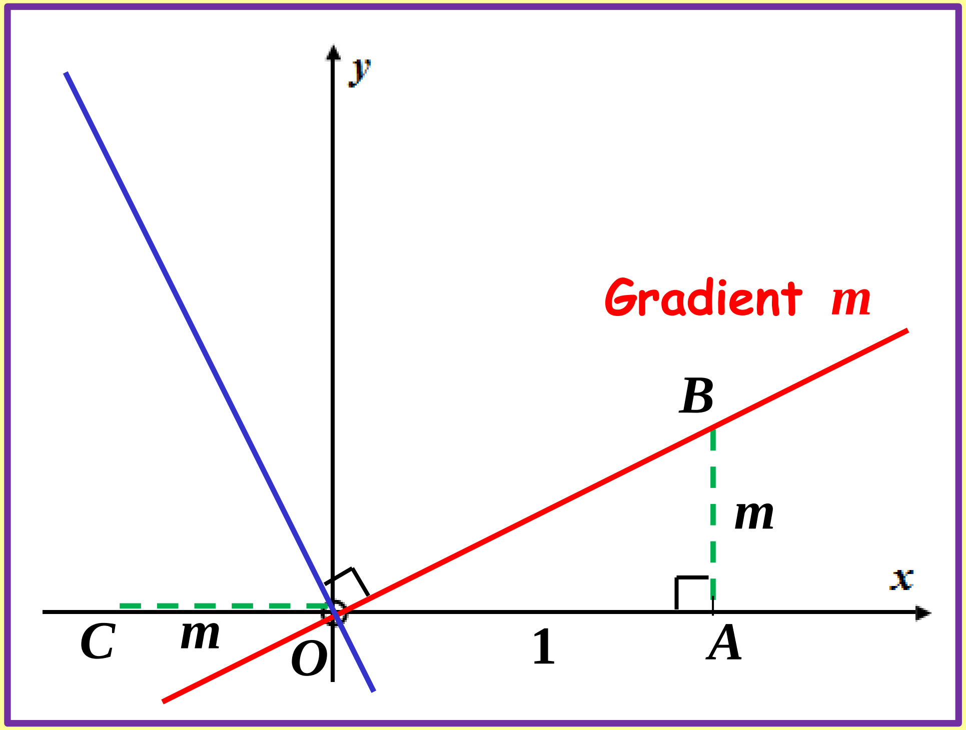
Proof that $m_2 = -\frac{1}{m_1}$ where m_1 and m_2 are the gradients of perpendicular lines.

Draw a line through O with gradient m

Mark A on the x-axis so that $OA = 1$

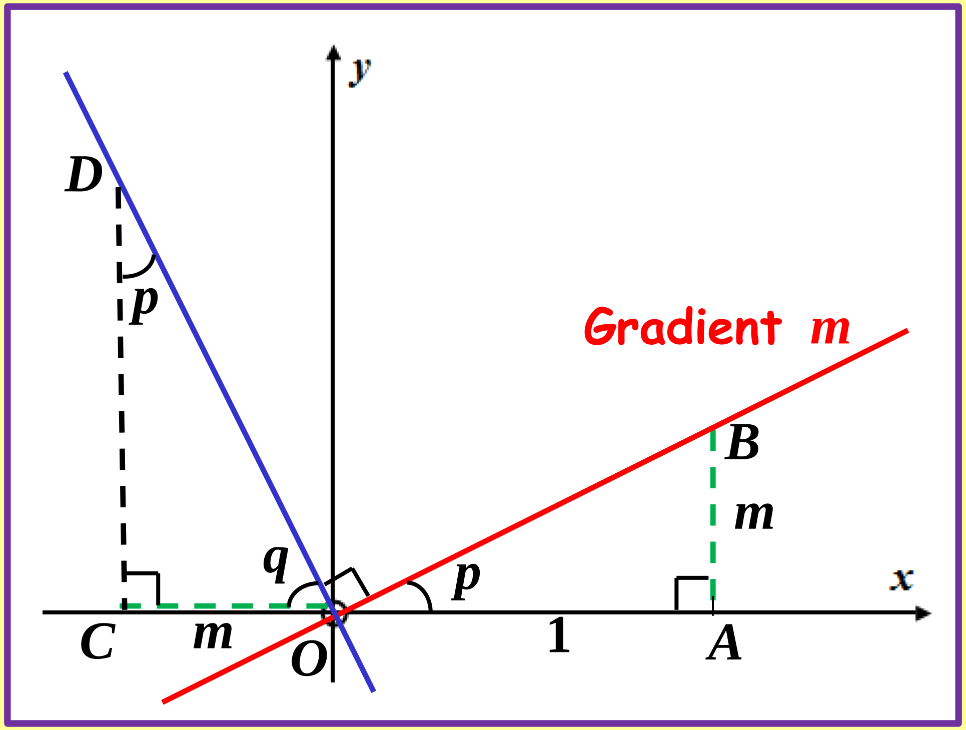
Then, $AB = m$

Draw the perpendicular to OB through O . . .



. . . and mark C on the x-axis so $OC = m$

Straight Lines and Gradients



Join CD

Let $\angle AOB = p$

and $\angle COD = q$

Then $q = 180° - 90° - p$

(Angles on a straight line)

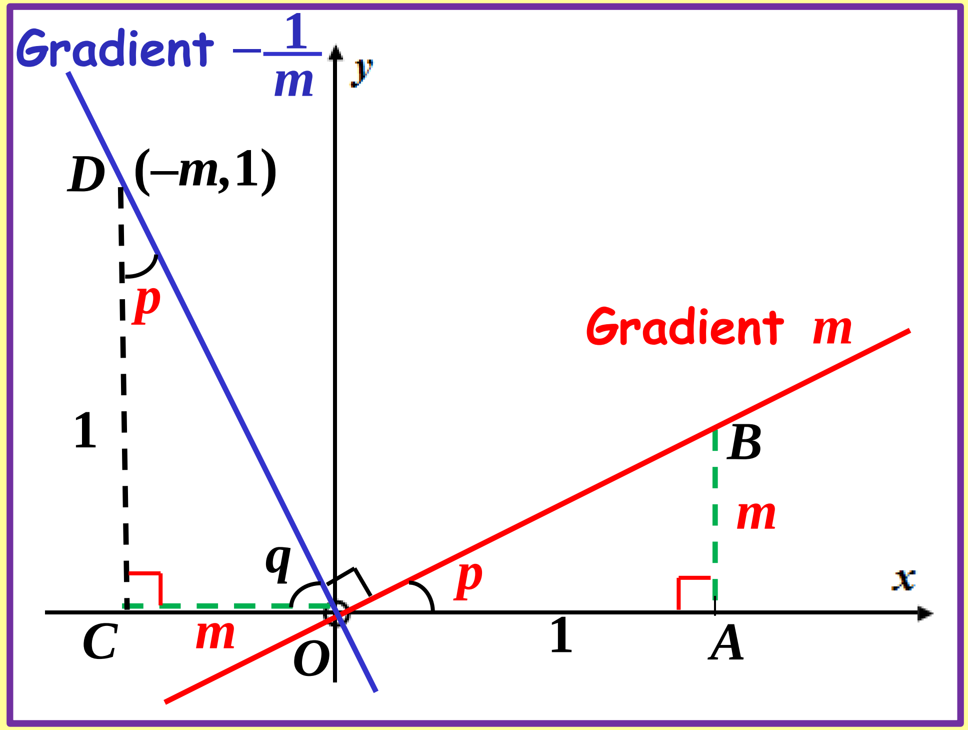
$\Rightarrow \quad q = 90° - p$

$\Rightarrow \quad \angle CDO = p$

(3rd angle of triangle OCD)

$\Rightarrow$ Triangles $\begin{matrix} OAB \\ DCO \end{matrix}$ are congruent (angle, angle, side)

Straight Lines and Gradients



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(3rd angle of triangle OCD)

$\Rightarrow$ Triangles $\begin{matrix} OAB \\ DCO \end{matrix}$ are congruent (angle, angle, side)

$\Rightarrow \quad DC = OA \quad \Rightarrow \quad DC = 1$

The coordinates of are $D(-m, 1)$

Gradient of $OD = \dfrac{1-0}{-m-0} \Rightarrow$ Gradient $= -\dfrac{1}{m}$

So the result is proved.

Straight Lines and Gradients

Straight Lines and Gradients

The following slides contain repeats of information on earlier slides, shown without colour, so that they can be printed and photocopied.

For most purposes the slides can be printed as "Handouts" with up to 6 slides per sheet.

Straight Lines and Gradients

- **Equation of a straight line**

$$y = mx + c \quad \text{or} \quad y - y_1 = m(x - x_1)$$

where m is the gradient and c is the intercept on the y-axis

- **Gradient of a straight line** $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

where (x_1, y_1) and (x_2, y_2) are points on the line.

If 2 lines have gradients m_1 and m_2 then:

- They are parallel if $m_2 = m_1$
- They are perpendicular if $m_2 = -\dfrac{1}{m_1}$

The perpendicular formula can be written as

$$m_1 \times m_2 = -1$$

Straight Lines and Gradients

e.g. Find the equation of the line through the points $(2, -3)$ and $(-1, 3)$.

Solution: $y - y_1 = m(x - x_1)$ We need m.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$ where $x_1 = 2$ and $y_1 = -3$.

$y_2 = -1$ and $y_2 = 3$.

$$m = \frac{3 - (-3)}{-1 - 2} \Rightarrow m = \frac{6}{-3} \Rightarrow m = -2$$

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - (-3) = -2(x - 2)$$

$$\Rightarrow y + 3 = -2x + 4 \Rightarrow y = -2x + 1$$

We could use the 2nd point (-1, 3) instead of (2, -3).

Straight Lines and Gradients

e.g. Find the equation of the line perpendicular to $y = 2x + 1$ passing through the point $(1, 4)$

Solution: The given line has gradient 2. Let $m_1 = 2$

Perpendicular lines: $m_2 = -\frac{1}{m_1} \Rightarrow m_2 = -\frac{1}{2}$

$$y - y_1 = m(x - x_1) \Rightarrow y - y_1 = -\frac{1}{2}(x - x_1)$$

$(1, 4)$ on the line $\Rightarrow y - 4 = -\frac{1}{2}(x - 1)$

Multiply by 2: $\Rightarrow 2y - 8 = -(x - 1)$

$$\Rightarrow 2y - 8 = -x + 1$$

$$\Rightarrow 2y = -x + 9$$